

M3 SOLUTION OF QUESTION PAPER

(CBCGS DEC 2017)

Q.1] a) Find Laplace transform of $f(t) = te^{-3t} \sin t$. (5)

Solution:

$$L[\sin t] = \frac{1}{s^2 + 1^2}$$

$$\therefore L[t \cdot \sin t] = (-1)^1 \frac{d}{ds} \left(\frac{1}{s^2 + 1} \right)$$

$$= -1 \times \frac{-1}{(s^2 + 1)^2} \times 2s$$

$$= \frac{2s}{(s^2 + 1)^2}$$

$$\therefore L[te^{-3t} \sin t] = \frac{2(s + 3)}{[(s + 3)^2 + 1]^2} \text{ (First shifting Method)}$$

$$= \frac{2(s + 3)}{[s^2 + 6s + 9 + 1]^2}$$

$$= \frac{2(s + 3)}{(s^2 + 6s + 10)^2}$$

$$\therefore L[te^{-3t} \sin t] = \frac{2(s + 3)}{(s^2 + 6s + 9)^2}$$

b) Obtain Complex form of Fourier series of $f(x) = e^x, -1 < x < 1$ in $(-1, 1)$. (5)

Solution:

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Let $c = -1$ and $c + 2l = 1$

$$\therefore -1 + 2l = 1$$

$$\therefore 2l = 2$$

$$\therefore l = 1$$

Given, $f(x) = e^x$

$$C_n = \frac{1}{2l} \int_c^{c+2l} f(x) e^{-inx/l} dx$$

$$= \frac{1}{2(1)} \int_{-1}^1 e^x e^{-inx/1} dx$$

$$= \frac{1}{2} \int_{-1}^1 e^{x-in\pi x} dx$$

$$= \frac{1}{2} \int_{-1}^1 e^{(1-in\pi)x} dx$$

$$= \frac{1}{2} \left[\frac{e^{(1-in\pi)x}}{1-in\pi} \right]_{-1}^1$$

$$= \frac{1}{2} \times \frac{1}{1-in\pi} [e^{(1-in\pi)} - e^{-(1-in\pi)}]$$

$$= \frac{1}{2(1-in\pi)} \times [e^1 e^{-in\pi} - e^{-1} e^{1-in\pi}] \times \frac{(1+in\pi)}{(1+in\pi)}$$

Consider,

$$e^{\pm in\pi} = \cos n\pi \pm i \sin n\pi = (-1)^n \pm i0 = (-1)^n$$

$$\therefore C_n = \frac{(1+in\pi)}{2(1^2 - i^2 n^2 \pi^2)} \times [e^1 (-1)^n - e^{-1} (-1)^n]$$

$$= \frac{(1+in\pi)}{2(1+n^2\pi^2)} (-1)^n \times [e^1 - e^{-1}]$$

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$$= \frac{(1+in\pi)}{1+n^2\pi^2} (-1)^n \times \sin h1$$

In Complex Fourier series,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/1}$$

$$\therefore e^x = \sum_{n=-\infty}^{\infty} \frac{(1+in\pi x)}{(1+n^2\pi^2)} \sin h1 \cdot e^{in\pi x/1}$$

$$\therefore e^x = \sin h1 \sum_{n=-\infty}^{\infty} \frac{(1+in\pi x)}{(1+n^2\pi^2)} (-1)^n \cdot e^{in\pi x/1}$$

c) Does there exist an analytic function whose real part is $u = k(1 + \cos \theta)$? Give justification. (5)

Solution:

$$u = k(1 + \cos \theta)$$

Differentiating partially w.r.t. r' , $\frac{\partial u}{\partial r} = 0 \rightarrow (1)$

Again, differentiating partially w.r.t. r' , $\frac{\partial^2 u}{\partial r^2} = 0 \rightarrow (2)$

Similarly, differentiating 'u' partially w.r.t. θ' ,

$$\frac{\partial u}{\partial \theta} = k(0 - \sin \theta) = -k \sin \theta$$

Again, differentiating partially w.r.t. θ' , $\frac{\partial^2 u}{\partial \theta^2} = -k \cos \theta \rightarrow (3)$

Consider Laplace's equation in polar form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} =$$

$$= 0 + \frac{1}{r} \cdot (0) + \frac{1}{r^2} \cdot -k \cos \theta \quad (\text{From 1,2 and 3})$$

$$= \frac{-k}{r^2} \cos \theta$$

$$\neq 0$$

\therefore 'u' does not satisfy Laplace's equation.

\therefore 'u' is not harmonic.

So, u is not a part of analytic function $f(z) = u + iv$.

Hence, there does not exist an analytic function whose real part is

$$u = k(1 + \cos \theta).$$

d) The equations of lines are of regression are $3x + 2y = 26$ and $6x + y = 31$. Find 1) means of x and y, 2) coefficient of correlation between x and y. (5)

Solution:

$$3x + 2y = 26$$

$$\therefore y = -\frac{3}{2}x + \frac{26}{2} \rightarrow (1)$$

And, $6x + y = 31$

$$\therefore y = -6x + 31 \rightarrow (2)$$

$$\text{Let } b_1 = -\frac{3}{2} \text{ and } b_2 = -6$$

$$\text{Since } |b_1| < |b_2|, \quad b_{yx} = b_1 = -\frac{3}{2} \text{ and } b_{xy} = \frac{1}{b_2} = \frac{-1}{6} \rightarrow (3)$$

\therefore Equation (1) is regression equation of Y on X type and equation (2) is regression equation of X on Y type.

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From (1) and (2), $-\frac{3}{2}x + 13 = -6x + 31$

$$\therefore 6x - \frac{3}{2}x = 31 - 13$$

$$\therefore \frac{9}{2}x = 18$$

$$\therefore x = 4$$

Substituting $x = 4$ in (2), $y = -6(4) + 31$

$$\therefore y = 7$$

$$\text{Now, } r = \pm \sqrt{b_{yx} \cdot b_{xy}}$$

$$= \pm \sqrt{\frac{-3}{2} \times \frac{-1}{6}} \quad (\text{From 3})$$

$$= \pm \frac{1}{2}$$

Since, b_{yx} and b_{xy} are both negative, 'r' is negative.

$$\therefore r = -1/2$$

$$\text{Now, } b_{yx} = r \frac{\sigma_y}{\sigma_x}$$

$$\therefore \frac{-3}{2} = \frac{-1}{2} \times \frac{\sigma_y}{3}$$

$$\therefore \sigma_y = 9$$

Hence,

Means of x and y ; $\bar{x} = 4$; $\bar{y} = 7$

Coefficient of Correlation (r) between x and $y = 0.5$ $\sigma_y = 9$

Q. 2] a) Evaluate $\int_0^{\infty} e^t \sin 2t \cos 3t dt$. (6)

Solution:

$$\text{Consider, } L[\cos 3t \sin 2t] = \frac{1}{2} L[2 \cos 3t \sin 2t]$$

$$= \frac{1}{2} L[\sin(3t + 2t) - \sin(3t - 2t)]$$

$$\therefore L[\sin 2t \cdot \cos 3t] = \frac{1}{2} \{L[\sin 5t] - L[\sin t]\}$$

$$\therefore \int_0^{\infty} e^{-st} \sin 2t \cos 3t dt = \frac{1}{2} \left\{ \frac{5}{s^2 + 5^2} - \frac{1}{s^2 + 1^2} \right\}$$

Put $s = -1$

$$\therefore \int_0^{\infty} e^t \sin 2t \cos 3t dt = \frac{1}{2} \left\{ \frac{5}{(-1)^2 + 25} - \frac{1}{(-1)^2 + 1} \right\}$$

$$\therefore \int_0^{\infty} e^t \sin 2t \cos 3t dt = \frac{-2}{13}$$

b) Find the image of the square bounded by lines

$x = 0, x = 2, y = 0, y = 2$, in the z -plane under the transformation

$$w = (1 + i)z + (2 - i). \quad (6)$$

Solution:

$$\text{Consider, } w = (1 + i)z + (2 - i)$$

$$\therefore u + iv = (1 + i)(x + iy) + (2 - i), \text{ (we put } w = u + iv \text{ and } z = x + iy)$$

$$\therefore u + iv = x + i^2y + ix + iy + 2 - i$$

$$\therefore u + iv = (x - y + 2) + i(x + y - 1)$$

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Comparing real and imaginary part on both sides, $u = x - y + 2 \rightarrow$
(1)

and $v = x + y - 1 \rightarrow$ (2)

Adding (1) and (2), $u + v = 2x + 1 \rightarrow$ (3)

Subtract (2) from (1), $u - v = -2y + 3 \rightarrow$ (4)

Now, when $x = 0$, (which is $Y -$ axis in $Z -$ plane)

From (3), $u + v = 1$ (which is a line having slope = -1 in $W -$ plane)

When $x = 2$, (which is line parallel to $Y -$ axis in $Z -$ plane)

From (3), $u + v = 2(2) + 1$

$\therefore u + v = 5$ (which is a line having slope = -1 in $W -$ plane)

When $y = 0$, (which is $X -$ axis in $Z -$ plane)

From (4), $u - v = 3$ (which is a line having slope = 1 in $W -$ plane)

When $y = 2$, (which is line parallel to $X -$ axis in $Z -$ plane)

From (4), $u - v = -2(2) + 3$

$u - v = -1$ (which is a line having slope = 1 in $W -$ plane)

Given, $w = (1 + i)z + (2 - i)$

Let $w_1 = (1 + i)z$, which is "Rotation and Magnification"

transformation, in which the shape is preserved.

$\therefore w = w_1 + (2 - i)$, which is "Translation" transformation,

in which the shape is preserved.

So, the above four lines in $w -$ plane will intersect to form a square.

Hence, the square (bonded by $x = 0, x = 2, y = 0, y = 2$) in the $z -$ plane

is mapped onto the square (bonded by $u + v = 1, u + v = 5, u - v = 3,$

$u - v = -1$ in the $W -$ plane.

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c) Obtain Fourier series of $f(x) = |x|$ in $(-\pi, \pi)$. Hence,

deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$ (8)

Solution:

$$f(x) = |x|$$

$$\therefore f(-x) = |-x| = |x| = f(x)$$

$\therefore f(x)$ is even function.

$$\therefore b_n = 0$$

Here, $l = \pi$

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} |x| dx = \frac{2}{\pi} \int_0^{\pi} x dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$$

$$= \frac{1}{\pi} [\pi^2 - 0] \quad (2 \text{ get cancelled.})$$

$$= \pi$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

$$= \frac{2}{\pi} \int_0^{\pi} |x| \cos \frac{n\pi x}{\pi} dx$$

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$$= \frac{2}{\pi} \int_0^{\pi} x \cos nx dx \quad (\pi \text{ get cancelled.})$$

$$= \frac{2}{\pi} \left[x \cdot \frac{\sin nx}{n} - 1 \cdot \frac{-\cos nx}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(\pi \cdot \frac{\sin n\pi}{n} + \frac{\cos n\pi}{n^2} \right) - \left(0 + \frac{\cos 0}{n^2} \right) \right]$$

$$= \frac{2}{\pi} \left[0 + \frac{(-1)^n}{n^2} - 0 - \frac{1}{n^2} \right]$$

$$= \frac{2}{\pi n^2} [(-1)^n - 1]$$

In Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\therefore |x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} [(-1)^n - 1] \cos \frac{n\pi x}{\pi} + 0$$

$$\therefore |x| = \frac{\pi}{2} + \frac{2}{\pi} \left(\frac{-2 \cos x}{1^2} + 0 - \frac{2 \cos 3x}{3^2} + 0 - \frac{2 \cos 5x}{5^2} + \dots \right)$$

$$\therefore |x| = \frac{\pi}{2} - \frac{4}{\pi} \left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right) \rightarrow (1)$$

Deduction: Put $x = 0$ in (1)

$$\therefore 0 = \frac{\pi}{2} - \frac{4}{\pi} \left(\frac{\cos 0}{1^2} + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2} + \dots \right)$$

$$\therefore -\frac{\pi}{2} = -\frac{4}{\pi} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right)$$

$$\therefore \frac{-\pi}{2} \times \frac{\pi}{-4} = \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right)$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

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Q. 3] a) Find the inverse Laplace transform of

$$F(s) = \frac{s}{(s^2 + 9)(s^2 + 4)} \quad (6)$$

Solution:

$$L^{-1} \left[\frac{s}{(s^2 + 9)(s^2 + 4)} \right] = L^{-1} \left[\frac{s}{s^2 + 4} \times \frac{1}{s^2 + 9} \right]$$

$$\text{Let } \phi_1(s) = \frac{s}{s^2 + 4} ; \phi_2(s) = \frac{1}{s^2 + 9}$$

$$\therefore f_1(t) = L^{-1} \left[\frac{s}{s^2 + 2^2} \right] = \cos 2t \text{ and } f_2(t) = L^{-1} \left[\frac{1}{s^2 + 3^2} \right] = \frac{1}{3} \sin 3t$$

$$\text{By Convolution theorem, } L^{-1}[\phi_1(s)\phi_2(s)] = \int_0^t f_1(u)f_2(t-u)du$$

$$\therefore L^{-1} \left[\frac{s}{s^2 + 4} \times \frac{1}{s^2 + 9} \right] = \int_0^t \cos 2u \cdot \frac{1}{3} \sin 3(t-u)du \times \frac{2}{2}$$

$$= \frac{1}{3} \times \frac{1}{2} \int_0^t 2 \cos 2u \cdot \sin(3t-3u)du$$

$$= \frac{1}{6} \int_0^t [\sin(2u+3t-3u) - \sin(2u-3t+3u)] du$$

$$= \frac{1}{6} \int_0^t [\sin(-u+3t) - \sin(5u-3t)] du$$

$$= \frac{1}{6} \left[\frac{-\cos(-u+3t)}{-1} + \frac{\cos(5u-3t)}{5} \right]_0^t$$

$$= \frac{1}{6} \left\{ \left[\cos(-t+3t) + \frac{\cos(5t-3t)}{5} \right] - \left[\cos(0+3t) + \frac{\cos(0-3t)}{5} \right] \right\}$$

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$$\begin{aligned}
 &= \frac{1}{6} \left\{ \left[\cos 2t + \frac{\cos 2t}{5} \right] - \left[\cos 3t + \frac{\cos 3t}{5} \right] \right\} \\
 &= \frac{1}{6} \left\{ \frac{6}{5} \cos 2t - \frac{6}{5} \cos 3t \right\} \\
 &= \frac{1}{6} \times \frac{6}{5} \{ \cos 2t - \cos 3t \} \text{ hgjyfgjkhbkhk} \\
 \therefore L^{-1} \left[\frac{s}{s^2 + 4} \times \frac{1}{s^2 + 9} \right] &= \frac{1}{5} (\cos 2t - \cos 3t)
 \end{aligned}$$

**b) Solve $\frac{\partial^2 u}{\partial x^2} - 100 \frac{\partial u}{\partial t} = 0$ with $u(0, t) = 0, u(1, t) = 0,$
 $u(x, 0) = x(1 - x),$ taking $h = 0.1$ for three steps upto
 $t = 1.5$ by Bender – Schmidt method. (6)**

Solution:

We are given $h = 0.1$ and $a = 100$

$$k = \frac{a}{2} h^2 \quad \therefore k = \frac{100}{2} \cdot (0.1)^2 = 0.5$$

Since, $h = 0.1,$ and the x is 0 to 1. We divide x interval into 10 parts by taking $h = 0.1.$

We also divide the time interval into 3 parts by taking $k = 0.5$

$$t_0 = 0, t_1 = 0.5, t_2 = 1.0, t_3 = 1.5$$

By data, $u(0, t) = 0$

Hence, for all $x = 0$ and $t = 0, 0.5, 1.0, 1.5.$

$$u(0, t) = 0 \text{ for all } t.$$

$$\therefore \text{By data } u(1, t) = 0$$

Hence for all $x = 1$ and $t = 0, 0.5, 1.0, 1.5$

$$\therefore u(1, t) = 0 \text{ for all } t.$$

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Now $u(x, 0) = x(1 - x)$

We now calculate u for $t = 0$ and

$x = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$

\therefore when $x = 0, t = 0, u = 0(1 - 0) = 0$

when $x = 0.1, t = 0, u = 0.1(1 - 0.1) = 0.09$

when $x = 0.2, t = 0, u = 0.2(1 - 0.2) = 0.16$

when $x = 0.3, t = 0, u = 0.3(1 - 0.3) = 0.21$

when $x = 0.4, t = 0, u = 0.4(1 - 0.4) = 0.24$

when $x = 0.5, t = 0, u = 0.5(1 - 0.5) = 0.25$

when $x = 0.6, t = 0, u = 0.6(1 - 0.6) = 0.24$

when $x = 0.7, t = 0, u = 0.7(1 - 0.7) = 0.21$

when $x = 0.8, t = 0, u = 0.8(1 - 0.8) = 0.16$

when $x = 0.9, t = 0, u = 0.9(1 - 0.9) = 0.09$

when $x = 1.0, t = 0, u = 1.0(1 - 1.0) = 0$

t \ x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0	0.09	0.16	0.21	0.24	0.25	0.24	0.21	0.16	0.09	0
1	0	0.08	0.15	0.20	0.23	0.24	0.23	0.20	0.15	0.08	0
2	0	0.075	0.14	0.19	0.22	0.23	0.22	0.19	0.14	0.075	0
3	0	0.07	0.1325	0.18	0.21	0.22	0.21	0.18	0.1325	0.07	0

Used Bender Schmidt formula $c = \frac{a + b}{2}$ for remaining values.

c) Using Residue Theorem evaluate

(8)

$$1] \int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$$

Solution:

$$\text{Let } z = e^{i\theta} \quad \therefore dz = ie^{i\theta} \cdot d\theta = izd\theta \quad \therefore d\theta = \frac{dz}{iz}; \quad \cos\theta = \frac{z^2 + 1}{2z}$$

$$\therefore I = \int_C \frac{z}{5 + 4\left(\frac{z^2 + 1}{2z}\right)} \cdot \frac{dz}{iz}$$

$$I = \int_C \frac{z}{i(2z^2 + 5z + 2)} \cdot \frac{dz}{z}$$

Where C is the circle $|z| = 1$

Now, the poles are given by $(2z + 1)(z + 2) = 0$

$$\therefore z = -\frac{1}{2}, \quad z = -2$$

The pole $z = -\frac{1}{2}$ lies inside the unit circle and the pole $z = -2$ lies outside.

Now, Residue of $f(z)$ (at $z = -\frac{1}{2}$)

$$= \lim_{z \rightarrow -\frac{1}{2}} \left(z + \frac{1}{2}\right) \cdot \frac{z}{2[z + (1/2)](z + 2)i}$$

$$= \frac{2}{2[-(1/2) + 2]i} = \frac{2}{3i}$$

$$= 2\pi i \left(\frac{2}{3i}\right) = \frac{4\pi}{3}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta} = \frac{4\pi}{3}$$

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Q. 4] a) Solve by Crank – Nicholson simplified formula

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0, u(0, t) = 0, u(5, t) = 100, u(x, 0) = 20, \text{ taking } h = 1 \text{ for}$$

one – time step.

Solution:

Here we have $a = 1, h = 1$

To use Crank – Nicholson formula, we must have

$$k = ah^2 = 1 \times 1^2 = 1$$

The interval of x is 0 to 5. The subinterval is of size $h = 1$.

$$\therefore x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5$$

By data $u(0, t) = 0$

When $x = 0$ for all values of $t, u = 0$, when $t = 0, 1$.

By data $u(x, 0) = 20$

For all values of x , when $t = 0, x = 0, 1, 2, 3, 4, 5$.

By data $u(5, t) = 100$

When $x = 5$ and $t = 0, u = 100$ and when $x = 5, t = 1, u = 100$.

t \ x	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	u_1	u_2	u_3	u_4	100

Now, by Crank – Nicholson formula we calculate the remaining of

u_1, u_2, u_3, u_4 .

$$\text{We use } e = \frac{1}{4}(a + b + c + d)$$

$$u_1 = \frac{1}{4}(0 + 0 + 20 + u_2) \rightarrow (1)$$

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$$u_2 = \frac{1}{4}(20 + u_1 + 20 + u_3) \rightarrow (2)$$

$$u_3 = \frac{1}{4}(20 + u_2 + 20 + u_4) \rightarrow (3)$$

$$u_4 = \frac{1}{4}(20 + u_3 + 100 + 100) \rightarrow (4)$$

By simplifying the following equation,

Thus, the final table is

t \ x	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	9.80	2.19	30.72	59.92	100

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b) Obtain the Taylor's and Laurent's series which represent the

function $f(z) = \frac{2}{(z-1)(z-2)}$ in the regions

(1) $|z| < 1$, (2) $1 < |z| < 2$. (6)

Solution:

$$\text{Let } \frac{2}{(z-1)(z-2)} = \frac{a}{z-1} + \frac{b}{z-2} \quad \therefore 2 = a(z-2) + b(z-1)$$

$$\text{When } z = 1, \quad 2 = -a \quad \therefore a = -2$$

$$\text{When } z = 2, \quad 2 = b$$

$$\therefore \frac{2}{(z-1)(z-2)} = \frac{-2}{z-1} + \frac{2}{z-2}$$

Case(1): When $|z| < 1$, clearly $|z| < 2$

$$\therefore f(z) = \frac{2}{1-z} - \frac{2}{2[1-(1-z/2)]} = 2[1-z]^{-1} - [1-z/2]^{-1}$$

$$\therefore f(z) = 2[1+z+z^2+z^3+\dots] - \left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots\right]$$

Case(2): When $1 < |z| < 2$, we write

$$\begin{aligned} \frac{2}{(z-1)(z-2)} &= \frac{-2}{z-1} + \frac{2}{z-2} \text{ as} \\ &= -\frac{2}{z[1-(1-z)]} - \frac{2}{2[1-(1-z/2)]} \\ &= -\frac{2}{z} [1-(1/z)]^{-1} - [1-(z/2)]^{-1} \end{aligned}$$

$$f(z) = -\frac{2}{z} \left[1 + \frac{1}{z} + \frac{1}{z^2} + \dots\right] - \left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \dots\right]$$

c) Solve $(D^2 - 3D + 2)y = 4e^{2t}$ with $y(0) = -3$ and $y'(0) = 5$ where

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$$D = \frac{d}{dt}. \tag{8}$$

Solution:

Let $L(y) = \bar{y}$. Then, taking Laplace transform,

$$L(y'') - 3L(y') + 2L(y) = 4L(e^{2t})$$

But $L(y') = s\bar{y} - y(0) = s\bar{y} + 3$

and $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} + 3s - 5$

∴ The equation becomes,

$$(s^2\bar{y} + 3s - 5) - 3(s\bar{y} + 3) + 2\bar{y} = 4\frac{1}{s-2}$$

$$(s^2 - 3s + 2)\bar{y} = \frac{4}{s-2} + 14 - 3s = \frac{-3s^2 + 20s - 24}{s-2}$$

$$\therefore \bar{y} = \frac{-3s^2 + 20s - 24}{(s-2)(s^2 - 3s + 2)} = \frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2}$$

By artial fraction, $\bar{y} = -\frac{7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

Taking inverse Laplace transform,

$$y = -7L^{-1}\left(\frac{1}{s-1}\right) + 4L^{-1}\left(\frac{1}{s-2}\right) + 4L^{-1}\left(\frac{1}{(s-2)^2}\right)$$

$$= -7cL^{-1}\frac{1}{s} + 4e^{2t}L^{-1}\frac{1}{s} + 4e^{2t}L^{-1}\frac{1}{s^2}$$

Hence, $y = -7e^t + 42e^t + 4te^{2t}$.

Q. 5] a) Find an analytical function $f(z) = u + iv$, if

$$u = e^{-x}\{(x^2 - y^2) \cos y + 2xy \sin y\}. \tag{6}$$

Solution:

Let $u = e^{-x}\{(x^2 - y^2) \cos y + 2xy \sin y\}$

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$$\begin{aligned}\therefore u_x &= -e^{-x}\{(x^2 - y^2) \cos y + 2xy \sin y\} + e^{-x}\{2x \cos y + 2y \sin y\} \\ &= e^{-x}[-(x^2 - y^2) \cos y + 2x \cos y + 2y \sin y - 2xy \sin y]\end{aligned}$$

$$u_y = e^{-x}[-(x^2 - y^2) \sin y - 2y \cos y + 2x \sin y - 2xy \cos y]$$

$$\therefore \varphi_1 = u_x, \text{ and } \varphi_2 = u_y$$

By Milne – Thompson method

$$f'(z) = \varphi_1(z, 0) - i\varphi_2(z, 0) = e^{-z}[-z^2 + 2z]$$

$$\therefore f(z) = \int e^{-z}[-z^2 + 2z]dz$$

Integrating by parts,

$$\begin{aligned}f(z) &= (-z^2 + 2z)(e^{-z}) - \int (-e^{-z})(-2z + 2)dz \\ &= e^{-z}(z^2 - 2z) + \int e^{-z}(2 - 2z)dz\end{aligned}$$

Integrating by parts again,

$$\begin{aligned}\therefore f(z) &= e^{-z}(c - 2z) + (2 - 2z)(-e^{-z}) - \int (-e^{-z})(-2)dz \\ &= e^{-z}(z^2 - 2z) - e^{-z}(2 - 2z) + 2e^{-z} \\ &= e^{-z}e^{-z} + c.\end{aligned}$$

b) Find the Laplace transform of

$$f(t) = t\sqrt{1 + \sin t}. \tag{6}$$

Solution:

$$\begin{aligned}\sqrt{1 + \sin t} &= \sqrt{[\sin^2(t/2) + \cos^2(t/2) + 2 \sin(t/2) \cos(t/2)]} \\ &= \sqrt{[\sin(t/2) + \cos(t/2)]^2} = \sin(t/2) + \cos(t/2)\end{aligned}$$

$$\therefore \sqrt{1 + \sin t} = L[\sin(t/2) + \cos(t/2)]$$

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$$= \frac{1/2}{s^2 + (1/2)^2} + \frac{s}{s^2 + (1/2)^2}$$

$$= \frac{1}{2} \cdot \frac{4}{(4s^2 + 1)} + \frac{4s}{(4s^2 + 1)}$$

$$= \frac{4s + 2}{(4s^2 + 1)} = \frac{2(2s + 1)}{(4s^2 + 1)}$$

$$\therefore L[\sqrt{1 + \sin t}] = -\frac{d}{ds} \left[\frac{2(2s + 1)}{(4s^2 + 1)} \right] = -2 \left[\frac{(4s^2 + 1)2 - (2s + 1)8s}{(4s^2 + 1)^2} \right]$$

$$= -2 \frac{[-8s^2 - 8s + 2]}{(4s^2 + 1)^2}$$

$$= 4 \frac{(4s^2 + 4s - 1)}{(4s^2 + 1)^2}$$

c) Obtain half range Fourier cosinr series of $f(x) = x, 0 < x < 2$.

Using Parseval's identity, deduce that –

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots \quad (8)$$

Solution:

Let $f(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right)$. Here, $l = 2$.

$$\therefore a_0 = \frac{1}{l} \int_0^l f(x) dx = \frac{1}{2} \int_0^2 x dx = \frac{1}{2} \left[\frac{x^2}{2} \right]_0^2 = 1$$

$$= \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx = \frac{2}{2} \int_0^2 x \cos \frac{n\pi x}{2} dx$$

$$= \left[x \frac{\sin(n\pi x/2)}{n\pi/2} + \frac{\cos(n\pi x/2)}{n^2 \pi^2 / 2^2} \cdot 1 \right]_0^2$$

$$\therefore a_0 = \left[2 \cdot (0) + \frac{\cos n\pi}{n^2 \pi^2 / 2^2} - 0 - \frac{1}{n^2 \pi^2 / 2^2} \right]$$

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$$= \frac{[(-1)^n - 1]}{n^2 \pi^2 / 2^2}$$

$$= \begin{cases} -4 \cdot \frac{2}{n^2 \pi^2} & \text{if } n \text{ is odd} \end{cases}$$

$$= \{0 \quad \text{if } n \text{ is even}$$

$$\therefore x = 1 - \frac{8}{\pi^2} \left[\frac{1}{1^2} \cos \frac{\pi x}{2} + \frac{1}{3^2} \cos \frac{3\pi x}{2} + \dots \right]$$

By Parseval's identity

$$\frac{1}{l} \int_0^l [f(x)]^2 dx = \frac{1}{2} [2a_0^2 + a_1^2 + a_2^2 + \dots]$$

$$\therefore l.h.s. = \frac{1}{2} \int_0^2 x^2 dx = \frac{1}{2} \left[\frac{x^3}{3} \right]_0^2 = \frac{4}{3}$$

$$\therefore \frac{4}{3} = \frac{1}{2} \left[2 + \frac{64}{\pi^4} \left\{ \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots \right\} \right]$$

$$\frac{8}{2} - 2 = \frac{64}{\pi^4} \left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots \right]$$

$$\therefore \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$$

Q. 6] a) If $f(z) = \oint_C \frac{3z^2 + 2z + 1}{z - a} dz$, $C: x^2 + y^2 = 4$ find the values

of $f(3)$, $f'(1 - i)$ and $f''(1 - i)$. (6)

Solution:

The circle $x^2 + y^2 = 4$ is $|z| = 2$.

1] The point $z = 3$ lies outside the circle $|z| = 2$.

\therefore By Cauchy's Integral Theorem

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$$f(3) = \oint_c \frac{3z^2+2z+1}{z-a} dz = 0$$

2] The point $z = -1$ i. e. $(1, -1)$ lies outside the circle. Hence, we take

$$\varphi(z) = 3z^2 + 2z + 1 \text{ which is analytical everywhere}$$

$$\therefore \int_c \frac{\varphi(z)}{z-a} dz = 2\pi i \varphi(a) = 2\pi i(3a^2 + 2a + 1)$$

$$\therefore f(\varphi) = \int_c \frac{3z^2 + 2z + 1}{z-a} dz = 2\pi i(3a^2 + 2a + 1)$$

$$\therefore f'(a) = 2\pi i(6z + 2) \text{ and } f''(a) = 2\pi i(6)$$

$$\therefore f'(1-i) = 2\pi i[6(1-i) + 2] = 2\pi i(8 - 6i)$$

$$\text{and } f''(1-i) = 12\pi i.$$

b) Find the coefficient of correlation between height of father and height of son from the following data,

Height of father: 65 66 67 68 69 71 73

Height of son : 67 68 64 68 72 69 70. (6)

Solution:

Let x and y denote height of father and height of son respectively.

x	y	$u_i = x_i - 68$	$v_i = y_i - 68$	u_i^2	y_i^2	$u_i v_i$
65	67	-3	-1	9	1	3
66	68	-2	0	4	0	0
67	64	-1	-4	1	16	4
68	68	0	0	0	0	0
69	72	1	4	1	16	4
71	69	3	1	9	1	3
73	70	5	2	25	4	1
	<i>Total</i>	3	2	49	38	24

Hence, $n = 7$

Karl Pearson's coefficient of correlation

$$r_{x,y} = r_{u,v} = \frac{n \sum uv - \sum u \sum v}{\sqrt{n \sum u^2 - (\sum u)^2} \sqrt{n \sum v^2 - (\sum v)^2}}$$

$$= \frac{7(24) - (3)(2)}{\sqrt{7(49) - (3)^2} \sqrt{7(38) - (2)^2}}$$

$$= \frac{162}{\sqrt{334} \sqrt{262}}$$

$$= 0.5476$$

Hence, Coefficient of Correlation between height of father and height of son (r) = 0.5476

c) A tightly stretched string with fixed end points $x = 0$ and $x = l$, in the shape defined by $y = kx(l - x)$ where k is a constant is released from this position of rest. Find $y(x, t)$, the vertical

displacement if $\frac{\partial^2 y}{\partial t^2} = c \frac{\partial^2 y}{\partial x^2}$. (8)

Solution:

The vibration of a string is given by $\frac{\partial^2 y}{\partial t^2} = c \frac{\partial^2 y}{\partial x^2} \rightarrow (1)$

Since the vibration of a string is periodic the solution of (1) is of form

$y = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mct + c_4 \sin mct) \rightarrow (2)$

Given initial boundary conditions are:

1) When $x = 0, y = 0$ for all t i.e. one end A of the string remains fixed throughout the motion.

∴ We get from (2)

$$0 = (c_1 + 0)(c_3 \cos mct + c_4 \sin mct)$$

$$\therefore c_1 = 0$$

$$\therefore y = c_2 \sin mx (c_3 \cos mct + c_4 \sin mct) \rightarrow (3)$$

2) Now $\partial y / \partial t = 0$ when $t = 0$ i.e. initially the string is steady. From (3)

$$\frac{\partial y}{\partial t} = c_2 \sin mx \{c_3(-mc) \sin mct + c_4(mc) \cos mct\}$$

Putting $t = 0, \frac{\partial y}{\partial t} = 0, 0 = c_2 \sin mx (c_4 mc) \therefore c_2 c_4 mc = 0$

If $c_2 = 0$ then (3) will give a trivial solution $y = 0. \therefore c_4 = 0$

Thus, from (3), we get,

$$y = c_2 c_3 \sin mx \cos mct$$

$$y = c_5 \sin mx \cos mct \quad (\text{where } c_2 c_3 = c_5) \rightarrow (4)$$

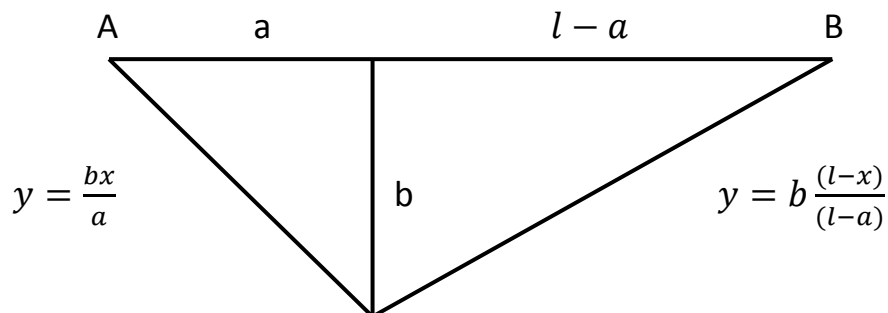
3) Now, $y = 0$ when $x = l$ for all t i.e. the other end of the string is fixed and the length of the string is l .

$$\therefore \text{From (4), we get } 0 = c_5 \sin ml \cos mct \rightarrow (5)$$

If $c_5 = 0$ (4) will lead to a trivial solution $y = 0$. Thus, c_5 cannot be zero.

$$\therefore \text{From (5), we get } \sin ml = 0$$

$$\therefore ml = n\pi \text{ i.e. } m = \frac{n\pi}{l}; \quad n = 1, 2, 3, \dots$$



As from above, the solution of given equation is

$$y = c_5 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \rightarrow (6)$$

Putting $n = 1, 2, 3, \dots$

$$y = \sum b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \rightarrow (7)$$

Now, lastly when $t = 0, y = kx(l - x)$

$$\therefore \text{Putting } t = 0 \text{ in (7) we get } y = \sum b_n \sin \frac{n\pi x}{l} \rightarrow (8)$$

where, y is given by $y = kx(1 - x)$.

But the series (8) is a Fourier half – range sine series for the function

The coefficient b_n can be determined from

$$\begin{aligned} b_n &= \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx = \frac{2}{l} \int_0^l kx(1 - x) \sin \frac{n\pi x}{l} dx \\ &= \frac{2k}{l} \left[x(1 - x) \cdot \left(-\frac{1}{n\pi} \cos \frac{n\pi x}{l} \right) - (1 - 2x) \left(-\frac{l^2}{n^2\pi^2} \right) + (-2) \left(\frac{l^3}{n^3\pi^3} \cos \frac{n\pi x}{l} \right) \right]_0^l \\ &= \frac{2k}{l} \left[-\frac{2l^3}{n^3\pi^3} \cos n\pi + \frac{2l^3}{n^3\pi^3} \right] = \frac{4kl^2}{n^3\pi^3} (1 - \cos n\pi) \end{aligned}$$

Hence, putting the value of b_n in (8) the solution is

$$y = \frac{4kl^2}{\pi^3} \sum \left(\frac{1 - \cos nx}{n^3} \right) \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}, \quad n = 1, 2, 3, \dots$$

$$\therefore y(x, t) = \frac{8kl^2}{\pi^3} \left[\frac{1}{1^3} \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} + \frac{1}{3^3} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l} + \dots \right]$$